4×4×4 Rubik's Cube God's Number: Notes on the OBTM ⩽ 53 and BT ⩽ 51 Upper Bounds

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Summary

The 4×4×4 cube has an OBTM upper bound of 53 (P1) and a BT upper bound of 51 (P3). The BT result improves the previously published bound of 53 (Rokicki 2015).

Metric	Previously published	This work	Method
OBTM upper bound	57 (Rokicki 2015)	53	17 + 16 + 20 three-stage decomposition, software-verified
BT upper bound	53 (Rokicki 2015)	51	Exact BFS (Colab) plus independent adversarial review
Lower bound (counting)	OBTM 35, BT 29	unchanged	n/a

- A GitHub repository (AnanasClassic/4x4x4-obtm-54-research, 2026) claims an OBTM bound of 54. It is unverified; cite it only as an unverified claim.
- A literature check found no newer competing bounds as of the check date. Re-confirm before presenting. A "55" figure attributed to Chen appears in project notes, but no primary source for it was found, so it is not listed here.

1. Setting and notation

Positions. A position is a sticker colouring of the 4×4×4 modulo whole-cube rotation. There are N = 7,401,196,841,564,901,869,874,093,974,498,574,336,000,000,000 positions. The cube has 24 centre slots, 24 wing slots and 8 corner slots.

Metrics.

- **OBTM** (outer block turn metric): a move turns the 1, 2 or 3 layers nearest one face by 90°, 180° or 270°.
- **BT** (block turn metric): a move turns any contiguous block of layers along one axis, except the whole cube. Per axis the blocks are [1], [2], [3], [4], [12], [23], [34], [123] and [234].
- Every OBTM move is a BT move, so d_BT ⩽ d_OBTM.

Earlier bounds. Rokicki (2015) obtained all three rows below from one three-stage reduction.

Metric	Stage 1	Stage 2	Stage 3	3×3×3	Upper bound
OBTM	8	14 − 1	16	20	57
SST	8	13	15	20	56
BT	7	12	15 − 1	20	53

The "− 1" is a saving at the junction between stage 2 and stage 3. The speedsolving wiki lists 29 ⩽ G ⩽ 53 for BT (page edited 2026-08-12).

Common structure of both proofs. Every position X is solved in three segments:

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X --(stages 1+2)--> Y in E3 --(stage 3)--> Z in R --(3×3×3 solve)--> solved
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- **E_3** is the stage-3 entry set: centres carry only the colours of their own axis pair, and the 24 wings can be paired up (details in §2.2).
- **R** is the stage-3 target: centres solved, all 12 wing pairs matched, parity bit 0.
- Stage 3 uses the group H3, generated by 20 moves: U, U2, U′, F, F2, F′, D, D2, D′, B, B2, B′, R2, L2, Uw2, Rw2, Fw2, Dw2, Lw2 and Bw2.
- The stage-3 coordinates are Edge3 × Center3, with 239,500,800 × 29,400 = 7,041,323,520,000 raw states. The target is (Edge3, Center3) = (0, 0).

2. OBTM ⩽ 53 proof (P1)

Theorem. Every position of the 4×4×4 cube is at OBTM distance at most 53 from solved.

The proof adds three bounds: 17 + 16 + 20 = 53.

Segment	What it does	Bound	Nature of the evidence
L1 (stages 1+2)	Reach E_3 from any position	17	Own computer-assisted proof
L2 (stage 3)	Reach R from E_3	16	Own exhaustive symmetry-reduced BFS
L3	Solve from R (collapses to a 3×3×3)	20	Cited theorem: 3×3×3 diameter 20 (Rokicki, Kociemba, Davidson, Dethridge, 2013)

2.1 L1: stages 1+2 reach E_3 in at most 17 moves (Theorem A)

The target set E_3. A position is in E_3 when:

- the centres carry only the colours of a single colour pair per face axis, with the centre pattern one of 6 target codes (the axis assignment is arbitrary);
- the 24 wings are arranged so that each wing pair has one wing in an A slot and one in a B slot (12 A slots, 12 B slots);
- the number s of flipped pairs is even, and the induced wing permutation π_W is even.

Sign lemma. sgn π_W = sgn R · (−1)^s, where R is the pairing permutation of §2.2. Hence on E_3 the permutation R is even, which is exactly what stage 3 can solve.

Stage 1 (centres). The coarse centre state space has Ω = 331,292,911,950 coordinates (Ω is the phase-1 coordinate count, not the full state count of about 7.4×10^{45}). Under the 48 cube symmetries it reduces to 6,902,491,710 classes, counted with Burnside's lemma. A breadth-first search over the quotient gives an exact distance table of diameter 12, which an independent Bellman-condition checker confirmed. Following strictly decreasing distances reaches the centre condition in d ⩽ 12 moves. This table, the depth-6 covariance of the good families and the symmetry-transport certificates come from the earlier r54 work. The depth-7 lemma ($B_7 = \varnothing$) was re-proved here.

Stage 2 (wings), budget 17 − d. The stage-1 coordinate remembers only centres, so the wing pairing P may be any of 23!! = 316,234,143,225 perfect matchings, together with any of the 6 centre targets. For each P:

- A **cut** assigns one wing of each pair to side A and the other to side B. There are 2^{12} = 4,096 cuts; a cut and its complement count as one, giving 2,048; the parity condition keeps half, so **exactly 1,024 legal cuts** belong to each P.
- Any two legal cuts of P are linked by double exchanges (swapping two pairs at once preserves parity).

Symmetry used in stage 2. The 48 symmetries are used only for stage 1. Stage-2 certificates are transported with a 16-element residual group R_{16} (8 elements preserving the A/B split, 8 swapping it). An endpoint in another orientation is first moved to the standard frame by a whole-cube rotation, which costs 0 moves.

Cases by stage-1 distance d.

d	Stage-2 budget	Method	Scale and checks
⩽ 10	⩾ 7	One depth-7 common family of 155,174 cuts meets a legal cut of every matching ($B_7 = \varnothing$)	All 316,234,143,225 matchings × 6 centre targets; 483/483 shards; two independent exhaustive searches plus a Python audit; the 155,174-cut family is the exact intersection of six independently reproduced depth-7 families (SHA-256 b422324a…7882a)
11	6	Row certificate by pigeonhole (below)	All 860,329,718 quotient states; 1,726 shards replayed by an independent checker, 0 errors
11, no 6-move witness	5 after a 12-move stage 1	12 + 5 = 17	24,290 rows
12	5	Row certificate CERT or UNSAT_PC	17,470 rows (835,104 raw states) re-checked independently: 10,751 CERT, 6,719 UNSAT_PC, 0 OPEN

Every case sums to at most 17, so Theorem A follows.

Pigeonhole lemma. Let G be the set of cuts that reach E_3 within the stage-2 budget, and let the number of cuts outside G that can still be legal candidates be R (the residual). If R < 1,024, every matching P has a legal cut inside G, because its 1,024 distinct legal cuts cannot all fit in R slots.

Row certification (d = 11, 12). For a representative r, the good cuts reachable from the endpoints of its shortest stage-1 words are pulled back to r′s frame and merged into U(r). The checker verifies that at most 1,023 cuts lie outside U(r). A representative's certificate transfers to its whole orbit under R_{16} (the residual is unchanged by the symmetry). A row with 1,024 or more residual cuts is not certified by this method; this does not mean the row fails, so it is treated by the 12 + 5 route or by UNSAT_PC.

UNSAT_PC (pair consistency). A matching P is bad if all 1,024 of its legal cuts lie in the residual. A bad P gives each of its cuts a support structure: exchanging two pairs gives another legal cut of P, which must also lie in the residual. The algorithm repeatedly deletes cuts that lack such support, in the manner of candidate elimination in Sudoku. It also

deletes a cut when its 12×12 table of candidate pairings has no perfect matching. It stops when nothing changes. If the remaining pair-consistent core is empty, no bad P exists. This is sound, because a real bad P always survives every round. The converse is not claimed; rows with a non-empty core (OPEN) would need direct enumeration inside the core, and there were none.

Status. L1 is a computer-assisted proof. Every row is closed, but no external party has independently recomputed the certificates. An adversarial GPT review accepted the argument on the reported audits; it did not rerun the code.

2.2 L2: stage 3 reaches R in at most 16 moves

Theorem L2. From every stage-3 state (Edge3, Center3), the 20 stage-3 moves reach (0, 0) in at most 16 moves, and some state needs exactly 16.

State space.

- **Edge3.** The 24 wing slots split into 12 fixed A slots and 12 fixed B slots. On E_3 each pair has one wing in an A slot and one in a B slot. Let R be the permutation that says which B-slot wing is the partner of the wing in each A slot. R is taken up to renaming of the pairs, and R = identity means every pair is matched. Because R is always even on E_3 (sign lemma), Edge3 has 12!/2 = 239,500,800 values.
- **Center3.** The remaining centre arrangement and the parity bit form a coordinate with 29,400 values (35 × 35 × 12 × 2; the meaning of these four factors follows the definition used by the TPR solver and is not interpreted here). All 29,400 values were enumerated and audited against the physical cube.
- **Parity bit.** It equals (centre term) ⊕ (wing-arrangement parity) ⊕ (corner parity). It is a physical invariant of all 20 stage-3 moves, and it is 0 on the target.
- **Size.** 239,500,800 × 29,400 = 7,041,323,520,000 states; the target is the single state (0, 0).

The 20 moves. U, F, D, B each by 90°, 180° and 270° (12 moves); R2 and L2 (2 moves); and the six two-layer half turns Uw2, Rw2, Fw2, Dw2, Lw2, Bw2 (6 moves). Each counts as one OBTM move.

Property P5. Each of the 20 moves maps the set of A slots to itself and the set of B slots to itself. If a move acts on A slots by α and on B slots by β, then $R \mapsto \beta R \alpha^{-1}$. So the next Edge3 value depends only on the current one and the move. This was verified from the actual wing-slot permutation of each of the 20 moves. Renaming pairs commutes with every move, which was also checked on 2,000,000 random transitions.

Symmetry reduction. There are 16 symmetries that fix the move set (as a set, with inverses) and the target.

- The 239,500,800 edge arrangements fall into (239,500,800 + 485,440) / 16 = 14,999,140 classes (Burnside).
- A **cell** is a pair (class representative, centre value), giving 14,999,140 × 29,400 = **440,974,716,000 cells**. Classes with a non-trivial stabiliser number 59,862. Cells over-count true orbits (440,113,212,144) by 861,503,856, which can only make recorded distances too large, never too small, so the upper bound stays valid.
- Raw states are recovered as the sum of 16/|Stab| over cells, which gives exactly 7,041,323,520,000.

Search. Breadth-first search from the target cell with these rules: a layer is built only from the previous layer; a recorded value is never changed; for classes with a stabiliser, every stabiliser element g also receives the same depth at (representative, g·c) (duplicate closure). Depths 1–14 are built top-down (expand visited cells). Depths 15 and 16 are built bottom-up (an unvisited cell is at depth k + 1 if some neighbour is at depth k), which is valid because the move set is closed under inverses.

Why the result is an upper bound. By induction on k, a cell first reached in layer k + 1 is adjacent to layer k and was not reached earlier, so its distance is exactly k + 1. The layers 0 to 16 sum to 440,974,716,000, the total number of cells, so every cell has distance at most 16. Layer 16 is non-empty, so 16 is exact.

Checks.

- The 16 symmetries were verified to be automorphisms of the move graph, against an independent physical model: all 239,500,800 edge arrangements normalised, 3,832,012,800 (symmetry, edge) pairs, and 61,312,204,800 edge group-closure cases plus 7,526,400 centre cases, with 0 violations.
- The cell-layer sizes agree at all 17 depths with Chen′s published distribution (2013); the raw size of layer 16 equals Rokicki′s independent count (2015). Edge3 was also audited exhaustively: all 239,500,800 coordinates × 20 moves = 4,790,016,000 transitions, with 0 failures of move-equivariance and of target correctness (coordinate 0 ⟺ all wing pairs matched); the Edge3 move graph has diameter 13. The TPR solver′s full-state moves agree with an independent sticker kernel on 2,757 words.
- The raw size of layer 16 is 1,909,413,996, equal to Rokicki′s independent count; the layer has 119,610,148 cells.
- Twenty raw depth-16 states were each shown by IDA* (with the maximum of the exact Edge3 and Center3 distances as the heuristic) to need exactly 16 moves. These solvers belong to one IDA* family, so this is not independent evidence, and it rests on the correctness of that implementation.

- An independent re-implementation (a new BFS code rotation, Edge3 moves computed on the fly) reproduced all layers; the SHA-256 of its sorted depth-16 cell list is identical to the original list.
- Only the upper bound 16 is used in the proof of 53.

2.3 L3: from R to solved in at most 20 moves

Cited theorem (Rokicki, Kociemba, Davidson, Dethridge, SIAM J. Discrete Math. 27(2):1082–1105, 2013). The Cayley graph of the 3×3×3 group with the 18 face turns (fixed centres) has diameter 20. The proof partitions the group into cosets of H = ⟨U, D, R2, F2, L2, B2⟩ (|H| = 19,508,428,800; 2,217,093,120 cosets), reduces them with the 48 symmetries and inversion, and checks by computer that every coset is solved within 20 moves (about 35 CPU-years). This work does not repeat that computation; it relies on the published result.

Collapse map π. A state Z ∈ R has centres solved as 2×2 blocks and every wing pair matched. Read each 2×2 centre block as one centre and each matched wing pair as one edge. This gives a 3×3×3 state π(Z).

π(Z) is a legal 3×3×3 state. A 3×3×3 state is reachable exactly when (1) the corner twist sum is 0 mod 3, (2) the edge flip sum is even, and (3) the edge and corner permutations have equal parity.

1. **Corner twist.** Corners move exactly as on a 3×3×3, so the twist sum is always 0.
2. **Edge flip.** The flip of composite edge i depends only on the pair p in A slot i and its type t: flip_i = t ⊕ g(p, i), where g(p, i) = 1 exactly when the home layer class of p differs from that of position i (U/D layers, slots 0–7, versus the middle layer, slots 8–11). This formula was checked on all 12 × 12 × 2 = 288 local cases with an independent solver's legality test, with 0 violations. The number of pairs sent from U/D positions to middle positions equals the number sent the other way, so Σ g is even, and the flip sum is congruent to s (mod 2). The value s is invariant under stage-3 moves and even on entry to E_3, so the flip sum is even.
3. **Permutation parity.** The parity bit is c(α) ⊕ par(B) ⊕ par(cp), where par(FEM) = 1 gives eparity = par(B). When Edge3 = 0, A = B, so on R the bit is 0 exactly when par(A) ⊕ par(cp) = c(α), where c(α) is the centre term for the axis assignment α. The centre term c is not always 0 (see the corrections in §2.5). Its value is determined by the homomorphism χ = sgn β on the rotation group, where β is the permutation of face axes induced by the rotation. Because the bit is a physical invariant and is 0 on the target, the three conditions hold.

Lifting. The 18 face turns of the 3×3×3 commute with π, checked on 96 marked stickers generator by generator. Each 3×3×3 move therefore lifts to exactly one outer single-layer turn of the 4×4×4 that keeps the state inside R. Also checked: 90,000 outer-generator

lifts, 5,000 legal-3×3×3 projections and 5,000 independent min2phase replays.

So every $Z \in R$ is solved in at most 20 OBTM moves.

2.4 Composition

Take any position X. Use a fixed frame with the 36 OBTM moves.

1. By Theorem A there is a word w_1 with $|w_1| \leqslant 17$ and $Y = X \cdot w_1 \in E_3$ (a finishing rotation costs 0).
2. The stage-3 coordinates of Y lie in Edge3 × Center3.
3. By Theorem L2 there is a word w_2 with $|w_2| \leqslant 16$ and $Z = Y \cdot w_2 \in R$.
4. By §2.3, π(Z) is a legal 3×3×3 state, solved in at most 20 face turns, each lifting to one outer turn.

Total: 17 + 16 + 20 = 53. ∎

2.5 Corrections to the P1 manuscript (main theorem unaffected)

1. **§7(b).** The statement "the centre term c equals 0 whenever centres are solved" is false: c = 1 in 1,504 of 3,000 random target states, and all six axis assignments occur. The correct statement is Lemma B2*: bit = 0 ⟺ par(A) ⊕ par(cp) = c(α) ⟺ the collapsed 3×3×3 is legal in its own centre frame. The proof uses the homomorphism χ = sgn β on the rotation group. Bridge 2, and hence the theorem, is unaffected.
2. **§1.3.** E_3 requires each face axis to carry the colours of a single colour pair, not necessarily its own pair. The axis assignment α is arbitrary on E_3.
3. **Terminology.** Ω = 331,292,911,950 is the phase-1 coordinate count, not the full state count. The six target codes are not filtered by parity; they are inherently even. The wing condition is not part of the phase-1 target.

3. BT ⩽ 51 proof (P3)

Theorem. Every position of the 4×4×4 cube is at BT distance at most 51 from solved. The bound is 17 + 14 + 20.

3.1 Ingredients

(A) Theorem A (§2.1). For every position X there is a fixed-frame OBTM word w with $|w| \leqslant 17$ and a finishing rotation ρ (cost 0) such that $X \cdot w \cdot \rho \in E_3$. Every OBTM move is a BT move, so this carries over.

(B) BT stage 3. H3_BT is the 20 H3 moves together with 9 BT half turns:

- the inner single-slice half turns u2 = U2·Uw2, d2 = D2·Dw2, f2 = F2·Fw2, b2 = B2·Bw2, r2 = R2·Rw2 and l2 = L2·Lw2;
- the middle-block (layers 2–3) half turns U2·Uw2·D2·Dw2, F2·Fw2·B2·Bw2 and R2·Rw2·L2·Lw2.

Each extra generator is a single BT move. On the fixed-frame stickers it equals the stated product of H3 moves, checked on sticker geometry built independently of the H3 move tables; so it preserves E_3 and the frame. The set is closed under inverses (each extra generator is an involution) and under the 16 stage-3 symmetries (u2, d2, f2, b2 are permuted among themselves; the U/D and F/B middle blocks are swapped; the R/L middle block is fixed; r2 and l2 are swapped by mirrors). Write d_BT3 for the distance to R in its Cayley graph.

Theorem L2-BT (computed, §3.5). d_BT3 ⩽ 15 on all 7,041,323,520,000 stage-3 states. Exactly 56 raw states (5 cells) have d_BT3 = 15, and each is a strict maximum: all 29 of its neighbours have d_BT3 = 14. The depth distribution agrees with Rokicki's BT stage-3 column at all 16 depths.

(C) The 3×3×3 diameter (§2.3). Face turns are BT moves, so a collapsed position is solved in at most 20 BT moves.

3.2 Axis syllables

Fix an axis a. A word that turns only layers of axis a is described by its 4-layer vector $v \in (\mathbb{Z}/4)^4$. Here v_i is the net number of quarter turns of layer i: layer 1 is the named face (R, U or F) and layer 4 is the opposite face. Vectors differing by a multiple of (1,1,1,1) are the same element modulo a rotation about a.

- **c_O(v)** is the least number of OBTM moves on axis a realising v modulo such a rotation. Normalising layer 4 to 0, it is the number of nonzero coefficients of v on the blocks [1], [12] and [123].
- **c_BT(v)** is the least number of BT moves realising v. It is computed by breadth-first search over the 256 vectors with the 9 BT blocks, and c_BT ⩽ c_O everywhere.
- The **translation group** DONT PANIC_a is the set of vectors realised by H3 words on axis a. On the R/L axis it is the 16 vectors with all coordinates even. On the U/D and F/B axes it is the 64 vectors with v_2 and v_3 even. The 9 extra BT generators lie in these groups.

3.3 Lemma J (junction)

Let w be the word of (A) with |w| = 17, and Y = X·w·ρ ∈ E_3.

1. Conjugate ρ to the front: X·w·ρ = X·ρ·w^ρ. Rotations map OBTM and BT moves to moves of the same kind and length, and X·ρ is the same position as X, so assume ρ = id.

2. Write w = v·S, where S is the maximal final run on one axis a, with vector s. If c_O(s) < |S|, the word shortens to length ⩽ 16, giving 16 + 15 + 20 = 51. Otherwise c_O(s) = |S| ⩾ 1.
3. Let g ∈ 🚫_a. Moves on one axis commute, so X·v·(S·g) = Y·g.
4. S·g is realised by c_BT(s+g) BT moves, up to a rotation about a, producing the same position Y·g.
5. Since g ∈ H3, the state Y·g is in E_3.

Therefore, for every g ∈ 🚫_a:

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d_{BT}(X) \le 17 - c_O(s) + c_{BT}(s+g) + d_{BT3}(Y \cdot g) + 20
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3.4 Reduction lemma and composition

Reduction lemma (exhaustive over 3 axes × 256 vectors, recomputed independently). For every axis a and every s with c_O(s) ⩾ 1, at least one holds:

- **(i)** some g ∈ 🚫_a has c_BT(s+g) ⩽ c_O(s) − 1;
- **(ii)** some half-turn H3 generator g on axis a has c_BT(s+g) ⩽ c_O(s).

Only 24, 8 and 8 vectors (R/L, U/D, F/B axes) need case (ii). Up to rotation these are the quarter turns R, Rw, L, Lw, Uw, Dw, Fw, Bw and their inverses; each is swapped for the opposite quarter turn via R2, Rw2, Uw2 and so on at no extra cost. The lemma also holds when the alternative syllable is priced in OBTM.

Composition. Take X, w and Y as above.

- |w| ⩽ 16: the bound is 16 + 15 + 20 = 51.
- |w| = 17 and d_BT3(Y) ⩽ 14: the bound is 17 + 14 + 20 = 51.
- |w| = 17 and d_BT3(Y) = 15: apply Lemma J.
  - In case (i): 17 − c_O + (c_O − 1) + 15 + 20 = 51.
  - In case (ii): Y·g is a neighbour of the strict maximum Y, so d_BT3(Y·g) = 14, and 17 − c_O + c_O + 14 + 20 = 51.

After stage 3 the cube is in R. By Lemma B2* (§2.5) the collapsed position is a legal 3×3×3 position, solved in at most 20 face turns, each lifting to one outer turn. ∎

Attribution. The strict-maximum saving at the stage-2/stage-3 junction is Rokicki's (2015), stated for a final R/L quarter turn of his stage 2. The new parts are the first term 17 (in place of 7 + 12 = 19) and the extension of the saving to every final syllable on every axis, which the arbitrary words of Theorem A require.

3.5 Computation

The BT stage-3 search took 640 s on a host with 48 CPU threads and 176 GB RAM.

Method.

- **Symmetry tables.** The 16-element stage-3 symmetry tables were rebuilt with a parallel version of the OBTM construction. There are 14,999,140 edge classes; stabiliser sizes 1/2/4/8/16 occur for 14,939,278 / 58,290 / 1,450 / 110 / 12 classes. The SHA-256 digests of the class, index, representative, centre-symmetry and stabiliser tables equal the certified OBTM digests (5 of 5). An independent physical model agrees on 4,000,000 × 16 checks, with 0 violations.
- **Frozen-code BFS.** The search runs over 440,974,716,000 cells under the same rules as §2.2 (duplicate closure, compare-and-swap claims, layer separation). Each cell stores a 2-bit code meaning depth ⩽ 12, 13, 14 or ⩾ 15 (102.7 GiB). Layers 13 and 14 are computed bottom-up. A depth-15 cell is a code-3 cell with a code-2 neighbour; that neighbour is a direct witness that d ⩽ 15.

BT stage-3 distribution.

Depth	Cells	Raw states
0	1	1
1	2	3
2	21	49
3	161	600
4	1,241	7,814
5	11,370	102,809
6	118,119	1,348,606
7	1,343,991	17,868,420
8	16,060,630	234,452,146
9	193,946,480	2,974,523,970
10	2,292,495,217	35,959,010,450
11	23,951,494,782	380,135,222,472
12	161,124,758,056	2,571,726,262,872
13	247,678,669,058	3,958,934,429,976
14	5,715,816,866	91,340,289,756
15	5	56

Depth	Cells	Raw states
Total	440,974,716,000	7,041,323,520,000

The five depth-15 cells (index = k·29,400 + c) are 157383009481, 255342028684, 255342028698, 373639883881 and 373639883903. Their stabiliser sizes are 2, 1, 1, 2, 2, and each has all 29 neighbours at depth 14.

Checks.

- **Generator set.** An independent reviewer rebuilt the 29 generators from sticker geometry. A BFS to depth 5 gives 1, 3, 49, 600, 7,814, 102,809, equal to Rokicki's BT column. Without the middle-block half turns the counts are 1, 3, 46, 570, 6,935, so his BT includes the middle block. With H3 alone they are 1, 3, 34, 356, 3,568, the OBTM column.
- **Closure.** Closure under the 16 symmetries and under inverses was checked on all 29,400 centre coordinates and 200,000 edge coordinates; §3.1(B) gives the general argument.
- **Junction run.** All 56 raw states, on all 3 axes, against every syllable with c_O ⩾ 1: the maximum bound found is 51, with 0 violations, also when alternatives are priced in OBTM. The reduction lemma does not depend on this run.
- **Independent OBTM reproduction.** The same code with the 20 H3 generators reproduces Theorem L2 exactly: depths 0–14 equal the Chen and Rokicki distributions; code totals are 16,499,291,808 / 86,675,095,891 / 242,701,855,493 / 95,098,472,808; depth 16 has 119,610,148 cells and 1,909,413,996 raw states; and the SHA-256 of its sorted cell list matches the original list. This run took 454 s.

3.6 Open problems

1. A BT bound of 16 for stages 1+2, using inner slices in those stages, would give BT ⩽ 50.

4. Verification status

Claim	How it was checked	Status
L1: stages 1+2 ⩽ 17	Exhaustive computer proof over all matchings (d ⩽ 10) and row certificates (d = 11, 12); independent checker replays; 0 rows open	Computer-assisted; builds on r54 tables and certificates; not recomputed by an outside party

Claim	How it was checked	Status
L2: stage 3 ⩽ 16	Own symmetry-reduced BFS over 440,974,716,000 cells; all 17 cell-layer sizes match Chen (2013); depth-16 raw count matches Rokicki (2015); independent re-implementation matches; exact diameter 16 shown by IDA*	Strong
L3: 3×3×3 ⩽ 20	Cited published theorem; own checks on collapse legality (288 local cases, 90,000 lifts, 5,000 projections, 5,000 replays)	Rests on a peer-reviewed external result
BT stage 3 ⩽ 15	Exhaustive BFS; all 16 layer sizes match Rokicki's BT column; 5 depth-15 cells, 56 raw states	Strong
BT junction saving (Lemma J)	Reduction lemma recomputed independently; junction run over all 56 states × 3 axes × all syllables, 0 violations	Internal adversarial review; no outside expert review yet

5. References

1. T. Rokicki, H. Kociemba, M. Davidson, J. Dethridge. The diameter of the Rubik's Cube group is twenty. SIAM J. Discrete Math. 27(2):1082–1105, 2013.
2. T. Rokicki. 4x4x4 upper bounds: 57 OBTM confirmed; 56 SST and 53 BT calculated. Domain of the Cube Forum, node 541, 2015-03-03 (forum.cubeman.org/?q=node/view/541).
3. S. Chen. Solving the 4x4x4 in 57 moves (OBTM). Domain of the Cube Forum, node 525, 2013.
4. P1: The 4×4×4 cube can be solved in 53 outer block turns. Manuscript, 2026.
5. P3: 4×4×4 Cube in 51 Block Turns. Manuscript, 2026.
6. cs0x7f. TPR-4x4x4-Solver (commit e432528). GitHub.
7. Speedsolving wiki, "God's Algorithm" (edited 2026-08-12).
8. AnanasClassic/4x4x4-obtm-54-research. GitHub, 2026. Claims OBTM 54; unverified.