

3-Cycles

For the last layer (LL) wing edges on the 4X4X4 cube (and for each orbital of edges for large cubes in general), there are exactly 28 distinct cases for 3-cycles. When I say “distinct”, I am neglecting having the same case apart from a cube rotation about y.

To prove that there are exactly 28 distinct cases, apart from experimentation, I use combinations and permutations. Combinations are used to find all possible positions of three wing edges in the last layer, and permutations are used to figure out how pieces can be swapped with each other in those positions.

The positions

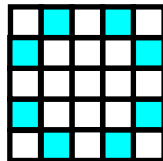
As I have previously said, I used combinations to find the exact number of different positions of three wing edges in the last layer.

For convenience, here is the formula for combinations for n objects taken k at a time :

$$C_{n,k} = \frac{P_{n,k}}{k!} = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

I initially thought that using 8 objects (because there are 8 wing edges in the last layer) and taking 3 at a time (3 wing edges involved in a 3-cycle) was the correct approach:

$$\binom{8}{3} = 56,$$



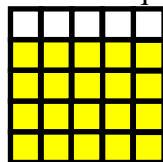
However, no matter what position the three wing edges are in, they will always be contained in 2 or 3 composite edges (not all four). Thus, computing $n=8$ would be unnecessary (and a lot more work). Since there is a pair of wing edges in each composite edge, then we should take $n = (2)(3) = 6$. Taking n to be 6 and k to be 3, the number of possible combinations is reduced to 20:

$$\binom{6}{3} = 20$$

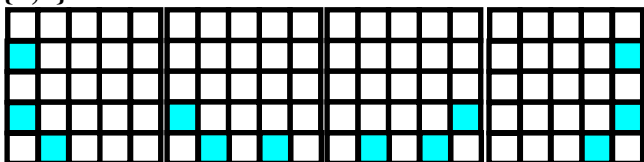
In each case, there are 3 wing edge pieces colored in cyan blue. A list with two numbers is written above each indicating the number of spaces between the 3 wing edges involved.

Every case will have one wing fixed in the top of the left composite, while the other two wing edges will move according to how many spaces are indicated in the list above that iteration.

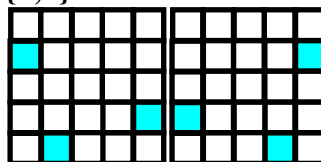
All iterations will be contained in the bottom region of the square (indicated by the yellow fill below).



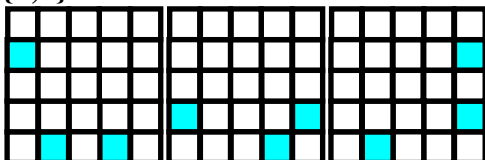
$\{0,0\}$



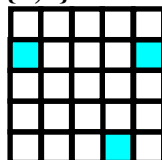
$\{1,1\}$



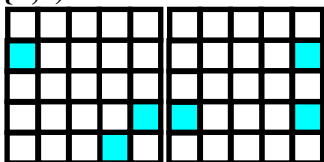
$\{1,0\}$



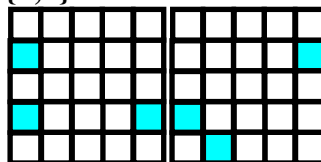
$\{2,1\}$



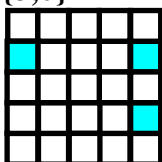
$\{2,0\}$



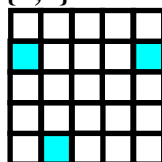
$\{0,2\}$



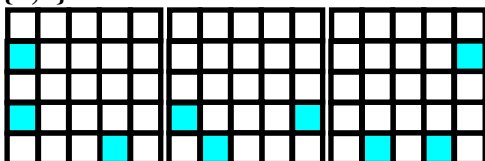
$\{3,0\}$



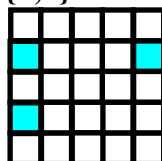
$\{1,2\}$



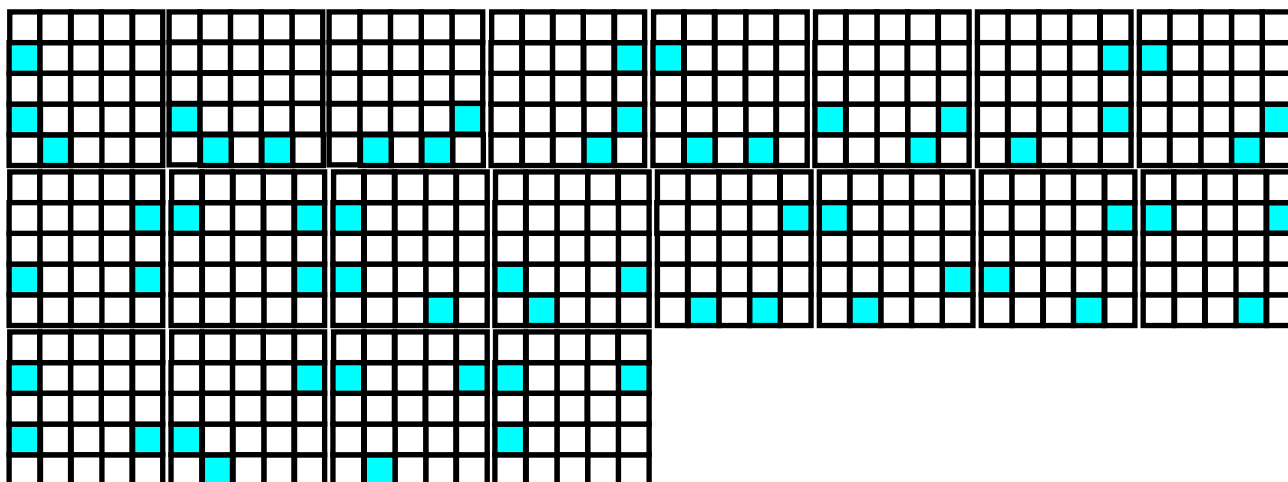
$\{0,1\}$



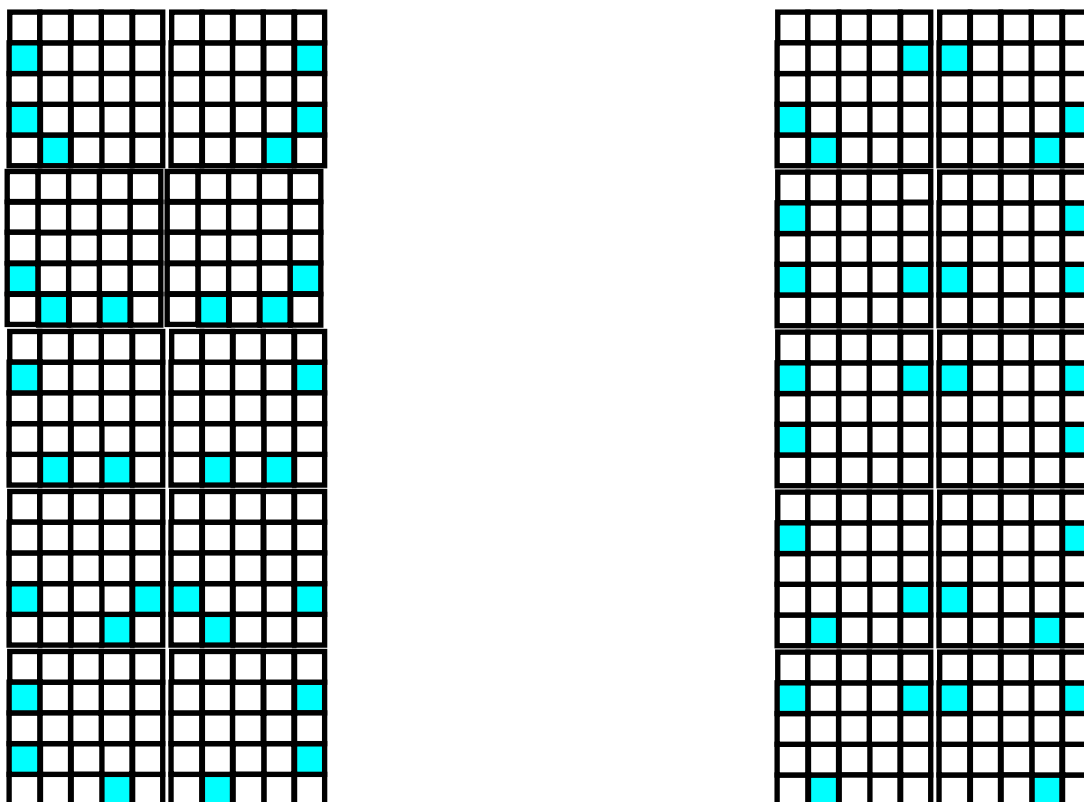
$\{0,3\}$



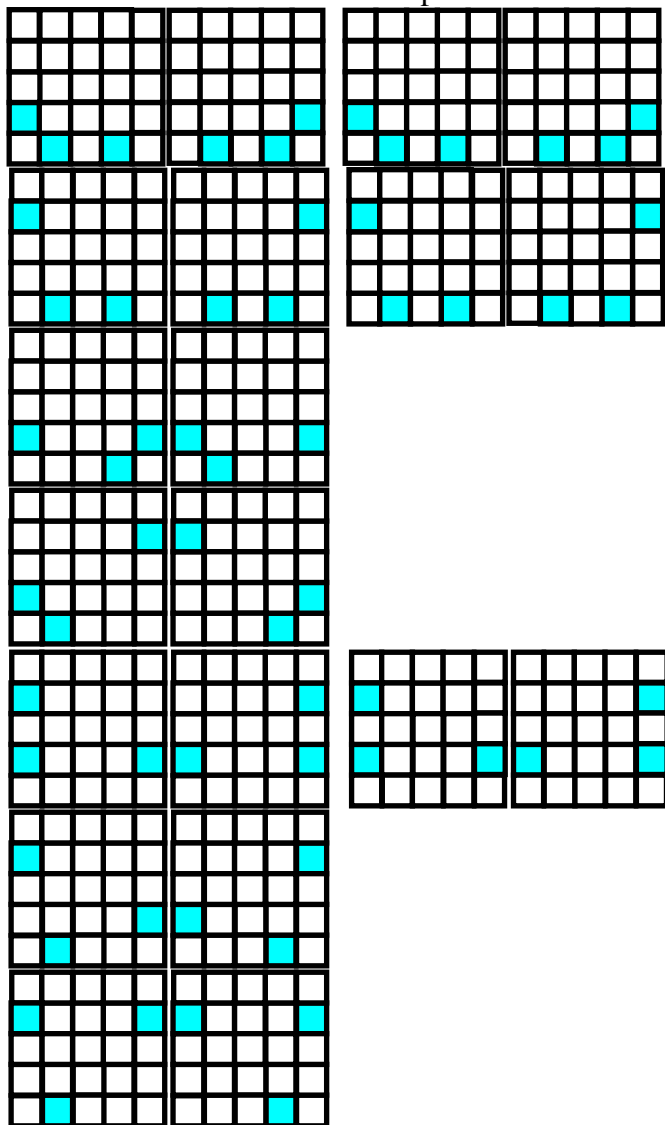
As you can see, all 20 combinations were found. I have put a copy of all 20 below to make them easier to see as a whole.



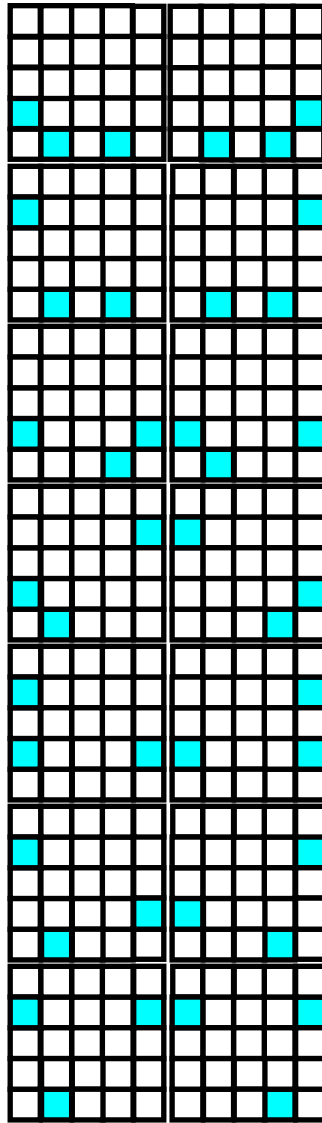
Next, I grouped each case with its mirror image.



I then rotated some of them and place them next to their equivalences.



Finally, I deleted the equivalent pieces, and we are left with:



Thus, it turns out that there are 14 different positions for 3 wing edges (on the same orbital) restricted in the last layer of a big cube.

Moving of the Pieces in the 14 Positions

There are only two ways to permute 3 wing edges all at once, either clockwise or counterclockwise. Therefore, there are a total of $2(14) = 28$ distinct cases. On the next page is a complete table of 3-cycles. The algorithms for each case are the ones which Thom Barlow provided on his website: <http://snk.digibase.ca/k4/>

Table of 3-Cycles

Scenario 1						
Clockwise (Let this set of algorithms be [A])						
$l2\ U'\ r'\ D2\ r$ $U\ r'\ D2\ r\ l2$	$r2\ l\ D2\ l'\ U'$ $l\ D2\ l'\ U\ r2$	$R2\ r'\ D'\ r$ $U2\ r'\ D\ r$ $U2\ R2$	$F\ L\ u\ L'\ U2$ $L\ u'\ L'\ U2$ F'	$L'\ U'\ L\ U\ l'$ $U'\ L'\ U\ (Ll)$	$l'\ R\ U\ R'\ U'$ $l\ U\ R\ U'\ R'$	$l'\ r'\ D2\ r\ U2$ $r'\ D2\ r\ U2\ l$
Counter clockwise ($[A]^{-1}$)						
$l2\ r'\ D2\ r\ U'$ $r'\ D2\ r\ U\ l2$	$r2\ U'\ l\ D2\ l'$ $U\ l\ D2\ l'\ r2$	$R2\ U2\ r'$ $D'\ r\ U2\ r'$ $D\ (Rr)\ R$	$F\ U2\ L\ u\ L'$ $U2\ L\ u'\ L'$ F'	$(Ll)'\ U'\ L\ U$ $l\ U'\ L'\ U\ L$	$R\ U\ R'\ U'\ l'$ $U\ R\ U'\ R'\ l$	$l'\ U2\ r'\ D2\ r$ $U2\ r'\ D2\ r\ l$
Scenario 2: Mirror of Scenario 1						
Clockwise (Let this set of algorithms be [B])						
$r2\ l\ D2\ l'\ U\ l$ $D2\ l'\ U'\ r2$	$l2\ U\ r'\ D2\ r$ $U'\ r'\ D2\ r\ l2$	$R2\ l\ D'\ l'$ $U2\ l\ D\ l'$ $U2\ R2$	$F'\ U2\ R'\ u'$ $R\ U2\ R'\ u$ $R\ F$	$(Rr)\ U\ R'$ $U'\ r'\ U\ R\ U'$ R'	$R'\ U'\ R\ U\ l$ $U'\ R'\ U\ R'\ l'$	$r\ U2\ l\ D2\ l'$ $U2\ l\ D2\ l'\ r'$
Counterclockwise ($[B]^{-1}$)						
$r2\ U\ l\ D2\ l'$ $U'\ l\ D2\ l'\ r2$	$l2\ r'\ D2\ r\ U$ $r'\ D2\ r\ U'\ l2$	$R2\ U2\ l\ D'$ $l'\ U2\ l\ D\ l'$ $R2$	$F'\ R'\ u'\ R$ $U2\ R'\ u\ R$ $U2\ F$	$R\ U\ R'\ U'\ r$ $U\ R\ U'$ $(Rr)'$	$l\ R'\ U'\ R\ U$ $l'\ U'\ R'\ U\ R$	$r\ l\ D2\ l'\ U2$ $l\ D2\ l'\ U2\ r'$

Table of 3-Cycles

Scenario 1						
Clockwise (Let this set of algorithms be [A])						
Counter clockwise ($[A]^{-1}$)						
Scenario 2: Mirror of Scenario 1						
Clockwise (Let this set of algorithms be [B])						
Counterclockwise ($[B]^{-1}$)						